PhD Preliminary Examination in Analysis

Department of Mathematics
New Mexico Tech

Syllabus

The PhD Preliminary Examination in Analysis is intended to determine whether a student has adequate knowledge in the general area of real and complex analysis to begin a research program in applied mathematics. The exam

6. Sequences and Series of Functions.
   Pointwise convergence, uniform convergence, Cauchy condition for

7. J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, McGraw-Hill, 2003

Note: Many previous editions of these books available on the market will su ce as well.

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Practice Exam

Real Analysis

1. (a) Let $I = [0,1]$ and $f,g : I \to \mathbb{R}$

11. Let $(a_n)_{n\in\mathbb{N}}$ be a real-valued sequence such that

$$a_1 \geq 0\,; \quad a_2 \geq 0\,; \qquad \text{and} \qquad a_{n+2} = (a_n a_{n+1})^{1/2} \quad \text{for } n \in \mathbb{N}.$$

(a) Show that (a_n) is convergent.

(b) Show that $\lim_{n\to\infty} a_n = (a_1 a_2^2)^{1/3}$.

12. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series and the series $\sum_{n=1}^{\infty} b_n$ is such that the series $\sum_{n=1}^{\infty}(b_n - b_{n+1})$ converges absolutely. Prove that the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

13. Let $A, B \subset \mathbb{R}$ be disjoint sets of real numbers, that is, $A \cap B = \emptyset$. Show that: i) If A is compact and B is closed, then there exists $\delta > 0$ such that $|a - b| > \delta$ for any $a \in A$ and for any $b \in B$; ii) If A is closed and B is closed, then the above assertion is false.

14. Let $A \subset \mathbb{R}$ be a closed set of real numbers. Prove that there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that the set of its zero points $F = \{x \in \mathbb{R} \mid f(x) = 0\}$ is precisely A, that is $F = A$.

15. Let $f : [0, \infty) \to \mathbb{R}$ be a function such that: i) f is continuous on $[0, \infty)$, ii) f is differentiable on $(0, \infty)$, iii) $f(0) = 0$, iv) $|f'(x)| \leq |f(x)|$ for all $x > 0$. Prove that $f(x) = 0$ for all $x \in [0, \infty)$.

16. Let I be an open interval and $f : I \to \mathbb{R}$ be a function differentiable on I. Prove that f' is continuous if and only if the inverse image under f' of any point is a closed set.

17. Let $I = [0, 1]$, $E \subset I$ be a countable subset of I, and $f : I \to \mathbb{R}$ be bounded on I and continuous on $I \setminus E$. Prove that f is Riemann integrable on I.

18. Let $I = [-1, 1]$ and $f : I \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right); & x \neq 0; \\ 0 & x = 0. \end{cases}$$

Determine whether f is Riemann integrable on I.

19. Let $I = [0, 1]$ and $f : I \to \mathbb{R}$ be a continuous function on I such that

$$\int_0^1 f(t)p(t)\,dt = 0$$

for any polynomial p. Prove that $f(x) = 0$ for any $x \in I$.

20. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, $\alpha \in \mathbb{R}$, and let A, B and C be the sets defined by $A = \{x \in \mathbb{R} \mid f(x) = \alpha\}$, $B = \{x \in \mathbb{R} \mid f(x) \geq \alpha\}$, and $A = \{x \in \mathbb{R} \mid f(x) < \alpha\}$. Show that the set3a694 -283i243 6.9738 cTJ/F14 9.9626 Tf 6.642 0 Td237 Ts0

21. Let $f : \mathbb{R} \to \mathbb{R}$. Prove that f is continuous if and only if the inverse image of any closed set is closed.

22. Prove that the intersection of an arbitrary collection of compact sets in $\mathbb{R}$ is compact.

Complex Analysis

1. Let $\omega : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be a function defined by

$$w(z) = z + \frac{1}{z}.$$

Let S be a set in the complex plane defined by

$$S = \left\{ z : \frac{1}{2} < |z| < 2 \right\}$$

Find and sketch the image $\omega(S)$ of the set S under the map ω.

2. Show that all five roots of the complex polynomial

$$P(z) = z^5 + 6z^3 + 2z + 10$$

lie in the annulus $1 < |z| < 3$.

3. Let C

8. Let f and g be entire functions with no zeros and having the ratio f/g equal to unity at infinity. Show that they are the same function. That is, if f and g are entire functions such that $f(z) \neq 0$ and $g(z) \neq 0$ for any z and $\lim_{z \to \infty} f/g = 1$, then $f = g$.

9. Let f be a complex function that is analytic and nonzero in a region D in a complex plane. Show that $|f|$ has a minimum value in D that occurs on the boundary of D.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(t)| \le e^{-a|t|}$ for any $t \in \mathbb{R}$, with some constant $a > 0$. Define a complex function F by

$$F(z) = \int_{-\infty}^{\infty} f(t) e^{izt} dt.$$

Find 26 Tf 8.5.9626 Tf dt :