- practice exam -
PhD Preliminary Exam in Numerical Analysis
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1. You are given a large linear least squares problem

$$\min \|Ax - b\|_2,$$

where A is a 50,000 by 2,000 matrix with about 0.1% nonzero entries. The condition number of A is around 100. The data in b are accurate to 5 digits. Discuss at least two approaches to solving this least squares problem. Compare your approaches in terms of the CPU time and storage required. How accurate would you expect the resulting solutions to be?

2. Consider the fixed point equation, $x = g(x)$, and the fixed point iteration

$$x_{i+1} = g(x_i)\,; \quad i = 0; 1; 2; \ldots$$

If the continuous function, $g(x)$, satisfies the Lipschitz condition

$$|g(x) - g(y)| \le \lambda |x - y|$$

in the closed interval, $I = [x_0 - \rho; x_0 + \rho]$, where $\rho > 0$, $0 \le \lambda < 1$, and x_0 satisfies

$$|x_0 - g(x_0)| \le (1 - \lambda)\rho$$

prove the following.

(a) All iterates, x_i, lie within the interval I, i.e. $x_0 - \rho \le x_i \le x_0 + \rho$.

(b) The iterates converge to a root, $s \in I$, i.e. $\lim_{i \to \infty} x_i = s$ and $s = g(s)$.

(c) The root, s, is unique.

(d) If $g \in C^2(I)$, $g'(s) = 0$, and $|g''(x)| \le M$, for all $x \in I$, then

$$|x_{i+1} - s| \le \frac{M}{2} |x_i - s|^2\,; \quad i = 0; 1; 2; \ldots$$

3. A matrix $B \in \mathbb{R}^{n \times n}$ is convergent if $\lim_{n \to \infty} B^n = 0 \in \mathbb{R}^{n \times n}$. The following three statements are equivalent.

   (i) B is convergent.

   (ii) $\lim_{n \to \infty} \|B^n\| = 0$ for some matrix norm.

   (iii) The spectral radius $\rho(B) < 1$.

   Assume that a matrix $A \in \mathbb{R}^{n \times n}$ is convergent. Prove the following statements using (i)-(iii):

   (a) $I - A$ is invertible.

   (b) $(I - A)^{-1} = I + A + A^2 + \cdots$

   (c) $\frac{1}{1+\|A\|} \leq \|(I - A)^{-1}\|$

   (d) If $\|A\| < 1$, then $\|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}$

4. Suppose that we apply the scheme

$$w_{i+1} = w_i + hf(t_i + h/2, w_i + (h/2)f(t_i, w_i))$$

   to solve the problem

$$y' = \lambda y, \quad y(0) = 1,$$

   where $\lambda < 0$. Write down an explicit formula for w_{i+1} in terms of w_i, λ, and h. Is the method stable for $\lambda = -30$ and $h = 0.1$? Based on your formula, for what values of λ and h will the method be stable?

5.

   (a) Give a definition of the induced matrix norm.

   (b) Prove that the induced norm is a matrix norm.

6. Chebyshev polynomials are defined by

$$T_n(x) = \cos(n \arccos x), \quad n = 0, 1, 2, \ldots .$$

   (a) Obtain a recurrence formula for computing $T_n(x)$.

   (b) Show that Chebyshev polynomials are orthogonal in the weighted inner product

$$\langle T_n, T_m \rangle = \int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1 - x^2}}\, dx, \quad n, m = 0, 1, 2, \ldots ,$$

   and determine $\langle T_n, T_n \rangle$.