Numerical Analysis Qualifying Exam
Mathematics Department, New Mexico Tech
Summer 2008

(Answer all 6 questions.)

1. Suppose that we want to approximate the definite integral $\int_{-1}^{1} f(x)dx$ using a linear combination of the function values $f(0)$, $f(1)$, and $f(2)$. That is,
$$\int_{-1}^{1} f(x)dx \approx af(-1) + bf(0) + cf(1).$$
Find coefficients a, b, and c so that the formula is exact for 0, 1st, and 2nd degree polynomials. Show that your formula is also exact for cubic polynomials. Derive an error term for your approximation.

2. Let a be some positive constant. It is possible to use Newton's method to calculate $x = 1/a$ without doing division. Using Newton's method, write down an iterative scheme for computing $1/a$ using only addition, subtraction, and multiplication. Specify a starting point x_0 for your iteration that ensures convergence.

3. Consider the initial-value problem
$$y' = f(t, y), \quad y(t_0) = \alpha.$$
   (a) Define the A-stability of a numerical method for the initial-value problem.
   (b) Find the region of A-stability for the implicit trapezoidal method
$$\begin{aligned} w_0 &= \alpha, \\ w_{j+1} &= w_j + \tfrac{h}{2}[f(t_j, w_j) + f(t_{j+1}, w_{j+1})] \end{aligned}$$
   and determine if the method is A-stable.

4. Let $A \in R^{n \times n}$ and $b \in R^n$.
   (a) Describe the Gauss-Seidel method for solving a linear system $Ax = b$.
   (b) Give the iteration matrix of the method.
   (c) Formulate the Gauss-Seidel method for the standard five-point finite difference approximation of the boundary value problem
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (x, y) \in \Omega = (0, 1) \times (0, 1), \quad u|_{\partial\Omega} = 0.$$

5. Let $g \in C[a,b]$ and $g : [a,b] \to [a,b]$.

   (a) Prove that g has a fixed point in $[a,b]$.

   (b) If, in addition, $g'(x)$ exists on (a,b) and $|g'(x)| \le k < 1$ for all $x \in (a,b)$, prove that the fixed point is unique.

6. Let Π_n be the set of polynomials of degree n or less.

   (a) Construct $p_i \in \Pi_i$, $i = 0, 1, 2$ such that $p_i(1) = 1$ and $\int_{-1}^{1} p_i(x) p_j(x)\, dx = 0$, when $i \neq j$.

   (b) Find the quadratic polynomial, $q_2(x)$, such that $\int_{-1}^{1} \left[x^3 - q_2(x) \right]^2 dx$ is minimal.