Numerical Analysis Qualifying Exam
Mathematics Department, New Mexico Tech
Spring 2009

(Answer all six problems)

1. Describe the solution of the least squares problem

$$\min_{x \in R^n} \|Ax - b\|_2$$

by the QR factorization method where $A \in R^{m \times n}$, $m > n$ and the rank of A is known.

2. Develop the steepest descent method for solving the linear system, $Ax = b$, with $A \in R^{n \times n}$, a positive definite matrix. Describe the selection of the search direction, the steplength, and discuss the update of the residual.

3. Let

$$f(x) = \frac{1 - \sin x}{\pi/2 - x}.$$

   (a) Explain why straight forward evaluation of $f(x)$ in double precision

(b) Show that the truncation error is given by

$$\tau_{i+1} = \frac{5}{12} y'''(\xi_i)\, h^2$$

where $\xi_i \in (t_i, t_{i+1})$.

6. Let $f(x) \in C^4[a,b]$. For any $y, z \in \mathbb{R}$, Simpson's rule is given by

$$S(y,z) = \frac{h}{3}\left[f(y) + 4f\left(\frac{y+z}{2}\right) + f(z) \right]$$

where $h = \frac{z-y}{2}$, and, in particular, satisfies

$$\int_a^b f(x)\,dx = S(a,b) - \frac{h^5}{90} f^{(4)}(\xi)$$

for some $\xi \in [a,b]$. The composite Simpson's rule satisfies

$$\int_a^b f(x)\,dx = S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right) - \frac{b-a}{180}\left(\frac{h}{2}\right)^4 f^{(4)}(\tilde{\xi})$$

for some $\tilde{\xi} \in [a,b]$, and where $h = \frac{b-a}{2}$.

(a) By equating the above relations and assuming that $f^{(4)}(\xi) = f^{(4)}(\tilde{\xi})$ derive an approximatation for the error

$$E(a,b) = \int_a^b f(x)\,dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)$$

involving $S(a,b)$, $S\left(a, \frac{a+b}{2}\right)$, and $S\left(\frac{a+b}{2}, b\right)$.

(b) Describe how an adaptive composite Simpson's algorithm can be developed from this error estimate to obtain an approximation to $\int_a^b f(x)\,dx$ to a desired accuracy of $\epsilon > 0$. You don't have to be specific about the algorithm, just describe how it would work in general.