Ph.D. Preliminary Examination in Numerical Analysis
Department of Mathematics
New Mexico Institute of Mining and Technology
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1. This exam is four hours long.
2. You need a scienti c calculator for this exam.
3. Work out all six problems.
4. Start the solution of each problem on a new page.
5. Number all of your pages.
6. Sign your name on the following line and put the total number of pages.
7. Use this sheet as a coversheet for your papers.

NAME: ______________________________________ No. of pages:__________

Problem 1. Consider the equation

$$x + \ln x = 0.$$

This equation has a solution somewhere near $x = 0.55$.

1. Derive a convergent fixed point iteration, $x_{n+1} = F(x_n)$ for solving this equation. Find an interval such that if x_0 is in this interval, the fixed point iteration will converge to the root.
2. Starting with $x_0 = 0.55$, use your iteration to solve the equation, obtaining a root accurate to 3 digits.

Problem 2. Describe the Taylor series method of order m for solving the initial value problem

3. You may use some required properties of positive definite matrices without giving their proofs.

Problem 5. Describe the main steps of Francis's Algorithm of degree one (also known as implicitly shifted QR algorithm) for computing eigenvalues and eigenvectors of a proper Hessenberg matrix $A \in \mathbb{C}^{n\times n}$. Describe how the Rayleigh and the Wilkinson shifts are selected. How is the convergence of the method determined with these shifts? A pseudocode of the algorithm is not required.

Problem 6. Consider the steepest descent method for solving a system of equations $Ax = b$, where A is symmetric and positive definite. In k-th iteration, let

$$r^{(k)} = b - Ax^{(k)}$$

be the residual vector. We update the solution with

$$x^{(k+1)} = x^{(k)} + tr^{(k)},$$

where

$$t = \frac{r^{(k)T} r^{(k)}}{r^{(k)T} A r^{(k)}}.$$

Show that $r^{(k+1)}$ is orthogonal to $r^{(k)}$.