Probability and Statistics, Sample Prelim Questions, Spring 2021

1. Let $X_1, X_2, \ldots, X_n$ be an independent random sample drawn from a Poisson distribution with mean , and > 0 is an unknown parameter.

$$f(x|\ ) = \begin{cases} e \quad \frac{x}{x!} & \text{for } x = 0, 1, 2 \ldots \\ 0 & \text{elsewhere} \end{cases}$$

Consider two estimators for , $T_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $T_2 = \frac{1}{n\ 1}\sum_{i=1}^{n}(X_i \quad X)^2.$

(a) Are T_1 and T_2 unbiased estimators for ? Justify your answer.

(b) Which estimator is more efficient for , T_1 or T_2? Justify your answer.

(c) Calculate the Cramer-Rao lower bound for unbiased estimator of $^2.$

(d) Suppose that has an exponential prior distribution with mean > 0,

$$f(\ |\ ) = \begin{cases} \frac{1}{\ }e^{-} & \text{for } \ > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Derive the posterior distribution of .

(e) Compute the posterior mean of and show that the posterior mean of is consistent for as $n \to \infty.$

2. Consider the following joint density for random variables X and Y:

$$f(x, y) = \begin{cases} k(1 \quad y), & \text{for } 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find k.

(b) Evaluate $E(X)$ and $E(X^2)$. Find the variance of X.

(c) Derive the conditional density $f(y|x)$ and the conditional expectation, $E[1 - Y|X]$. Hence or otherwise, evaluate $E(Y)$ and $Cov(X, Y)$.

(d) Find $P(Y < 2X).$

3. Suppose that $X_1; X_2; : : : ; X_n$ is an i.i.d. sample from a normal distribution with mean and variance 1. Remember that the density of X_i is $\frac{1}{\sqrt{2}} \exp\left(\ \frac{(x\quad)^2}{2}\right):$

(a) Show that the likelihood ratio test of the null hypothesis $\ = 3$ against the alternative hypothesis $\ = 5$ rejects the null hypothesis if and only if $X = \frac{\sum_{i=1}^{n} X_i}{n} > c$ for some c.

(b) Find the value of c if $n = 6$ and the probability of Type I error is 0.05.

4. Consider the linear regression model

$$y = X \quad + $$

where X is a matrix of size m by n and rank n, y is a vector of length m, is a vector of length n, and is a random vector with the multivariate normal distribution $N(0; \ ^2 I)$.

(a) Write down the likelihood function.

(b) Use the likelihood function from part (a) to Show that the MLE is

$$\hat{\ } = \arg\min kX \qquad yk_2^2$$

5. Consider the classical Gambler's Ruin Problem. Me and my friend are tossing a coin, if the coin comes up Heads, I win \$1 from my friend, if Tails, I lose \$1. I start with a dollars and my friend starts with b dollars. The game ends when one of us loses all their money. We will model the amount of money I have after ith toss as a Markov Chain X_i, for a nonnegative integer i, and $X_0 = a$. The state space is f0;2182Us8e6gBmTi/hF348 115negative intege

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6. Consider a Poisson process $X(t)$ with intensity , so that

$$p_k(t) = P(X(t) = k) = \exp(\quad t)\frac{(\; t)^k}{k!}:$$

Let W_k be the time when kth event happens.

(a) Using the formula for $p_k(t)$, derive an expression for the density of W_k.

(b) Calculate $E(W_3 \, j \, X(t) = 5)$ and $E(W_5 \, j \, X(t) = 3)$

7. Consider the single-server queue with i.i.d. Exponential (with the mean) interarrival and i.i.d. Exponential (with the mean) service times. Let $X(t)$ be the number of total customers (both under service and in queue) in the system at the time t. Model $X(t)$ as a life-and-death process. Find the limiting distribution $\;_k = \lim_{t \, ! \, 1} P(X(t) = k)$ and the conditions under which it exists.