Probability and Statistics, Sample Prelim IV Questions, Fall 2021

1. Let $X_1, \ldots, X_n$ be a random sample from a distribution with probability density function

$$f(x| \quad) = \frac{x \quad 1}{(\quad)} e^{\quad x/} ; \qquad > 0; \text{ with known } \quad = 2$$

   (a) Find the maximum likelihood estimator (MLE) of

   (b) Show that the method of moments estimator (MoM) of is the same as the MLE in part (a).

   (c) Let = 1/ . Find the MLE of (denote it as ^). Is ^ consistent? State the properties/results you are using when answering this question.

2. Let $X_1, \ldots, X_n$ be independent random variables where X_i Poisson( ), > 0 is unknown,

$$f(x| \quad) = \begin{cases} e \quad \frac{\quad x}{x!} & \text{for } x = 0, 1, 2 \ldots \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Determine the Cramer-Rao lower bound for the variance of unbiased estimators of .

   (b) Find the minimum variance unbiased estimator of . Also, nd the variance of this estimator.

   (c) Let the prior density for be exponential distribution with mean 1 (i.e., $f(\quad) = e$). Find an explicit expression for a Bayes estimate of .

3. Consider the following joint density for random variables X and Y:

$$f(x, y) = \begin{cases} kx, & \text{for } 0 < x, y < 1 \text{ and } x + y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Find k.

   (b) Find the marginal density of X. Are these random variables independent?

   (c) Find the conditional mean $E(X|Y)$.

   (d) Find $E(X)$ in two ways: rst, using the marginal in part (b), then, using the conditional in part (c).

4. The distribution of X given a parameter Y is Exponential with the rate Y, that is, $f(x|Y = y) = ye^{-yx}; x > 0$. Also, Y has a prior distribution that's Uniform on $[1; 2]$. Use conditioning to find $E(X)$ and $P(X > 1)$.

5. Consider a Poisson process $X(t$

7. Let $X(t)$ be a birth-and-death process with values in $\{0, 1, 2\}$, the initial value $X(0) = 0$ the birth rates $_0 = {}_1 = 1$ and the death rates $_1 = 2, {}_2 = 3$. Let $P_n(t) = P(X(t) = n)$. Find a system of differential equations for $P_n(t), n = 0, 1, 2$ and show that their solution is

$$\begin{cases} P_0(t) = 3/5 + \exp(\ 2t)/3 + \exp(\ 5t)/15 \\ P_1(t) = 3/10 \quad \exp(\ 2t)/6 \quad (2/15)\exp(\ 5t) \\ P_2(t) = 1 \quad P_0(t) \quad P_1(t) \end{cases}$$