Ph.D. Preliminary Examination in Numerical Analysis
Department of Mathematics
New Mexico Institute of Mining and Technology
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1. This exam is four hours long. It is closed-book and cheat sheets, notes and calculators are not allowed.
2. Work out all six problems.
3. Start solution of each problem on a new page.
4. Number all of your pages.
5. Sign your name on the following line and put the total number of pages.
6. Use this sheet as a cover-sheet for your papers.

NAME: ________________________________

No. of pages:___________

Problem 1. For the initial value problem

$$\frac{dy}{}$$

Problem 4.

a) Use Newton's method to derive an algorithm for computing the 5th root of a positive real number, a.

b) Show that your iteration will converge to $\sqrt[5]{a}$ from any starting point $x_0 > 0$.

Problem 5. Let A be a positive definite matrix. Consider a descent iterative method for solving a linear system $Ax = b$ such that, given an approximation $x^{(k)}$ and a nonzero search direction $p^{(k)}$, a new approximation $x^{(k+1)}$ is computed by

$$x^{(k+1)} = x^{(k)} + \alpha_k p^{(k)}$$

for some value of α_k. Let

$$J(y) = \frac{1}{2} y^T A y - y^T b.$$

a) Describe the exact line search method for finding α_k; that is, find α_k which is the unique solution of the minimization problem

$$J(x^{(k+1)}) = \min_{\alpha \in \mathbb{R}} J(x^{(k)} + \alpha p^{(k)}).$$

b) Let $r^{(k)} = b - Ax^{(k)}$ and $e^{(k)} = x^* - x^{(k)}$, where x^* is the exact solution of $Ax = b$, be the residual and the error vectors, respectively. Show that

$$r^{(k+1)} = r^{(k)} - \alpha_k A p^{(k)}.$$

Using this identity, prove that vector $r^{(k+1)}$ is orthogonal to both $r^{(k)}$ and $Ae^{(k)}$.

Problem 6. Consider a problem of stability of the evaluation of function f at point x. For a given absolute error h in x, the condition number of f at x can be defined as the ratio of the relative errors in $f(x)$ and x:

$$\text{cond}(f; h) = \frac{\frac{f(x+h) - f(x)}{f(x)}}{\frac{h}{x}}.$$

a) Assuming that $f'(x)$ exists, find

$$\lim_{h \to 0} \text{cond}(f; h).$$

The resulting formula is used to compute the condition number of a smooth function f.