Date: April, 2024 Name:

PhD Preliminary Examination in Analysis

Department of Mathematics
New Mexico Tech

2024

1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers defined by

$$x_1 > \sqrt{2} \quad \text{and} \quad x_{n+1} = \frac{2 + x_n}{1 + x_n}, \quad n = 1, 2, \ldots$$

Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ converges and find its limit.

2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $\alpha : [a, b] \to \mathbb{R}$ be an increasing function. Prove that f is integrable with respect to α on $[a, b]$.

3. Let $f : (0, 1) \to \mathbb{R}$ be a nonnegative smooth function such that it vanishes, $f(a) = f(b) = 0$, at two distinct interior points $a, b \in (0, 1)$. Prove that there exists a point $c \in (0, 1)$ such that $f'''(c) = 0$.

4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ for all $t \in \mathbb{R}$. Suppose that

$$f_1(t) \le f_2(t) \le f_3(t) \le \cdots \le f_n(t) \le \cdots, \quad \forall\, t \in \mathbb{R}.$$

Prove for every sequence $\{t_k\}_{k=1}^{\infty}$ of real numbers converging to a real number t, that is, if $\lim_{k\to\infty} t_k = t$, there holds

$$\liminf_{k\to\infty} f(t_k) \ge f(t).$$

5. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that if $\operatorname{Im} f(z) \le 0$ for any $z \in \mathbb{C}$ then f is constant.

6. Let $P(z)$ be a polynomial of degree higher than 2 and f be a meromorphic function defined by

$$f(z) = \frac{1}{P(z)}.$$

Show that the sum of the residues of f at all the zeros of P is equal to 0.

7. Let C be the a sufficiently small simple closed contour not passing through the origin oriented counterclockwise. Evaluate the integral

$$I_n = \frac{1}{2\pi i} \oint_C \frac{dz}{z} \left(z + \frac{1}{z} \right)^n,$$

where n is an integer. Consider the cases $n > 0, n = 0, n < 0$.

8. Let $\alpha, \beta \in \mathbb{R}$ be two real numbers. Use principal value integrals to show that

$$\int_0^\infty \frac{dx}{x^2}\left[\cos(\alpha x) - \cos(\beta x)\right] = -\frac{\pi}{2}\left(|\alpha| - |\beta|\right).$$